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ORDINARY HYBRID FINITE DIFFERENCE METHODS FOR SOLVING BURGERS' EQUATION.

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ABSTRACT

Burgers' equation appears as a model in turbulence and gas dynamics. We construct hybrid finite difference schemes from ordinary finite difference methods for solving this equation. Among the hybrid methods developed are the Crank-Nicholson-Du Fort and Frankel and Crank-Nicholson- Lax-Friendrich's and Du Fort and Frankel. We determine that the Du Fort and Frankel discretization have an improvement effect on other finite difference schemes whereas the Lax- Friedrich's method reduces their efficacy. We note that the Du Fort and Frankel method increases the number of grid points involved by one. The increase of the grid points is responsible for the improved accuracy of the Crank-Nicholson and the Hybrid Crank-Nicholson-Lax-Friedrich's, methods. The hybrid Crank-Nicholson-Lax-Friedrich's,-Du Fort and Frankel scheme is the most accurate.

Keywords: Burgers' Equation, Crank-Nicholson; Lax-Friedrichs'; Du Fort and Frankel methods.

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Introduction

The Burgers equation

$$u_t + \beta u u_x = \alpha u_{xx} \, (0 \le x \le 1) \times (t \ge 0)$$
$$u(x,0) = u_0(x)$$
………………………….……… (1.1)

can be solved by using ordinary finite difference methods (Ames[1], Jain[4], Mitchel & Griffiths[5]). The ordinary finite difference methods in the literature are mainly two: forward time central space (FTCS), and the Crank-Nicholson methods.

Let the numerical solution of the equation (1.1)

at the point $(x,t) = (mh, nk)$ be denoted by

$U_{m,n}$. At the point $(x, t+1) = (mh, (n+1)k)$

the FTCS approximation is given by

$$U_{m,n+1} = f\left(mh, nk, U_{m,n}, \frac{1}{h}\delta_x U_{m,n}, \frac{1}{h^2}\delta_x^2 U_{m,n}\right)$$
………………………….. (1.2)

The Crank-Nicholson method is given by

$$U_{m,n+1} = f\left(mh, nk, U_{m,n}, \frac{1}{h}\delta_x U_{m,n}, \frac{1}{2h^2}\delta_x^2\left(U_{m,n} + U_{m,n+1}\right)\right)$$
…………………………………… (1.3)

$$u(0,x) = u_0(x)$$

$$u_x(0, t) = p(x)$$
$$u_x(1, t) = q(x)$$

For Eqns. (1.2) and (1.3) the initial and boundary conditions must be taken into consideration. Finite difference approximation of the boundary conditions can also be made.

The scheme (1.2) is an explicit method while the scheme (1.3) is an implicit method. The methods stated above rarely make use of the boundary conditions.

Other similar methods have been cited in the literature. Drazin [3] discusses the scattering method for solving this equation.

The heat equation

$$u_t = \alpha u_{xx} \quad \text{…………………} \quad (1.4)$$

has been solved by forward time central space (FTCS),backward time central space (BTCS), Leap-frog, Du Fort and Frankel, and the Lax-Fredrich's methods [1,2,4,5,6,7,8].

In our paper we seek to blend the finite differences used in solving the heat equation and use them (the blended schemes) to solve the Burgers' equation (1.1). The blended methods are called hybrid methods. The hybrid methods we shall develop are the Crank-Nicholson –Lax-Fredrich's, Crank-Nicholson-Du Fort and Frankel and Crank-Nicholson- Lax-Fredrich's –

Numerical Schematics: Construction of hybrid methods

Pure Crank-Nicholson

It is necessary that first we develop the pure Crank-Nicholson method for solving equation (1.1) to be used for comparison. We shall then determine the effects of blending it with the other methods.

We want to discretize equation (2.1.2) basing on the Crank-Nicholson method. We have

$$\frac{\partial u}{\partial t} \approx \frac{U_{m,n+1} - U_{m,n}}{k} = \Delta_t U_{m,n}$$

.... (2.1.3)

where Δ_t is the forward difference operator with respect to t .

The discretization of $\frac{\partial}{\partial t}(\frac{u^2}{2})$ using the Crank-Nicholson method is given by

$$\beta \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \approx \frac{\beta}{2}\left[\frac{\partial}{\partial x}\left(\frac{U^2}{2}\right)_{m,n} + \frac{\partial}{\partial x}\left(\frac{U^2}{2}\right)_{m,n+1}\right]$$

………………………………………...(2.1.4)

Let $\frac{U^2}{2} = G$

Equation (2.1.4) then becomes

$$\beta \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \approx \frac{\beta}{2}\left[\frac{\partial}{\partial x}(G)_{m,n} + \frac{\partial}{\partial x}(G)_{m,n+1}\right]$$

.... (2.1.5)

By Taylor's expansion we have

$$\frac{\partial}{\partial x}(G)_{m,n+1} = \frac{\partial}{\partial x}(G)_{m,n} + k\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}(G)_{m,n}\right) + O(k^2)$$

.... (2.1.6)

Now

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}(G)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}(G)\right) = \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial U}\frac{\partial U}{\partial t}\right) = \frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial t}\right)$$

.... (2.1.7)

and so

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}(G)\right) \approx \frac{\partial}{\partial x}\left(U_{m,n}\Delta_t U_{m,n}\right)$$

…………………………………….. (2.1.8)

$$= \frac{U_{m+1,n}(U_{m+1,n+1} - U_{m+1,n}) - U_{m-1,n}(U_{m-1,n+1} - U_{m-1,n})}{2kh}$$

The discretized form of equation (2.1.6) is now

$$\left(\frac{\partial G}{\partial x}\right)_{m,n+1} = \frac{\left(U^2/2\right)_{m+1,n} - \left(U^2/2\right)_{m-1,n}}{2h} + k\left[\frac{U_{m+1,n}(U_{m+1,n+1} - U_{m+1,n}) - U_{m-1,n}(U_{m-1,n+1} - U_{m-1,n})}{2kh}\right]$$

……………………………..……. (2.1.9)

Equation (2.1.4) then becomes

$$\beta\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\approx\frac{\beta}{2}\left\{\begin{array}{l}\left[\dfrac{\left(U^2\right)_{m+1,n}-\left(U^2\right)_{m-1,n}}{2h}\right]\\ +k\left[\dfrac{U_{m+1,n}\left(U_{m+1,n+1}-U_{m+1,n}\right)-U_{m-1,n}\left(U_{m-1,n+1}-U_{m-1,n}\right)}{2kh}\right]\end{array}\right\}\quad(2.1.10)$$

given by

$$\frac{\partial u}{\partial t}+\beta uu_x\approx\Delta_t U_{m,n}+\frac{\beta}{2}\left\{\begin{array}{l}2\left[\dfrac{\left(U^2\right)_{m+1,n}-\left(U^2\right)_{m-1,n}}{2h}\right]\\ +k\left[\dfrac{U_{m+1,n}\left(U_{m+1,n+1}-U_{m+1,n}\right)-U_{m-1,n}\left(U_{m-1,n+1}-U_{m-1,n}\right)}{2kh}\right]\end{array}\right\}$$

$$=\Delta_t U_{m,n}+\frac{\beta}{4h}\left(U_{m+1,n}U_{m+1,n+1}-U_{m-1,n}U_{m-1,n+1}\right)\quad\ldots\ldots\ldots\ldots\quad(2.1.11)$$

For the term αu_{xx} the Crank-Nicholson method is given by $\frac{\alpha}{2h^2}(U_{m,n}+U_{m,n+1})$.

Thus the pure Crank-Nicholson scheme for solving (1.1) is

$$\Delta_t U_{m,n}+\frac{\beta}{4h}\left(U_{m+1,n}U_{m+1,n+1}-U_{m-1,n}U_{m-1,n+1}\right)=\frac{\alpha}{2h^2}\left(U_{m,n}+U_{m,n+1}\right)\quad\ldots\ldots\ldots\ldots\quad(2.1.12)$$

Hybrid Crank-Nicholson- Lax-Fredrich's Scheme

The term $U_{m,n}$ in $\Delta_t U_{m,n}$ is replaced by

$\frac{1}{2}(U_{m-1,n}+U_{m+1,n})$ in equation (2.1.12) to

obtain hybrid Crank-Nicholson- Lax-Fredrich's Scheme

The terms $U_{m,n}$ in $\Delta_t U_{m,n}$ and

$\frac{\alpha}{2h^2}(U_{m,n}+U_{m,n+1})$ in equation (2.1.12) are

replaced by

$\frac{1}{2}(U_{m,n-1}+U_{m,n+1})$ to obtain this hybrid

Hybrid Crank-Nicholson-Du Fort and Frankel Scheme

Crank-Nicholson-Du Fort and Frankel Scheme.

Hybrid Crank-Nicholson-Lax-Friedrich's-Du Fort and Frankel Scheme

The term $U_{m,n-1}$ in the left hand side of the scheme Hybrid Crank-Nicholson-Du Fort and Frankel mentioned in section 2.4 above is replaced by $\frac{1}{2}(U_{m-1,n-1} + U_{m+1,n+1})$ to

obtain the hybrid Crank-Nicholson-Lax-Friedrich's-Du Fort and Frankel Scheme.

Results from the Numerical Experiments

We note that Lax-Friedrich's and Du Fort and Frankel differencing modifies the pure Crank-Nicholson scheme. In particular the Du Fort and Frankel differencing increase the number of grid points involved by one. All the schemes developed are based on Crank-Nicholson method and therefore all are unconditionally stable.

Wood [9], gives the exact solution of Burgers' equation (1.1) as

$$u(x,\ t) = \frac{2\lambda\pi e^{-\pi^2\lambda t}\sin \pi x}{d + e^{-\pi^2\lambda t}\cos\pi x}, \quad d > 1 \ \ldots..(3.1)$$

and so

$$u(x,0) = \frac{2\lambda\pi \sin \pi x}{d + \cos\pi x},\ d > 1, \ldots\ldots\ldots\ldots\ldots.(3.2)$$

$$u(0,t) = u(1,t) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3.$$

Equations (3.2) and (3.3) gives the initial and boundary values respectively. We use them in generating the numerical solution of the problem (1.1). We generate the solutions of the methods developed for following data: $h = 0.1,\ k = 0.001,\ d = 2, \alpha = 0.0001$ and $\beta = 1$.

The choice of the above parameters is to ensure that that accuracy is improved and that

the condition given in Eqn. (3.1) is satisfied.

Figures 1-5 display the results obtained from the constructed schemes. The notations used in the figures are given in the appendix.

We present the graphical results of the Burgers' equation at $t = 0.005$. The graphical results from the various hybrid methods and the exact

solution have been plotted together for comparison.

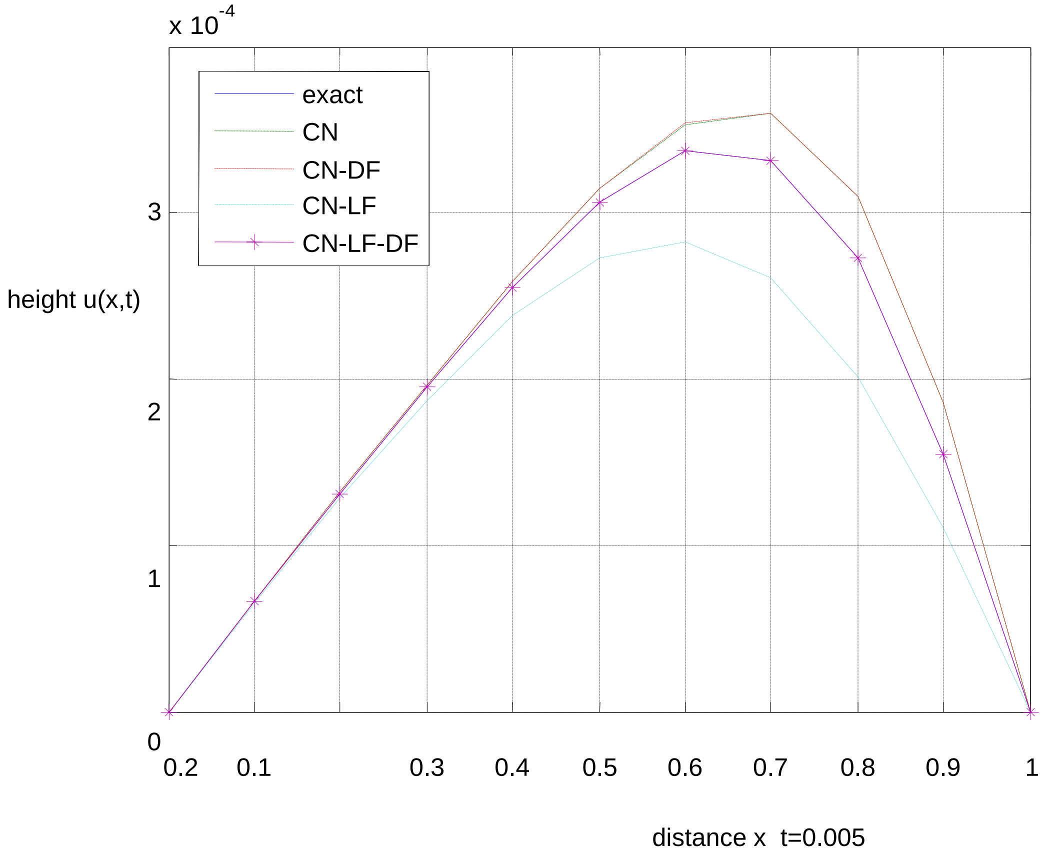


Figure 1: Solutions of the Burgers 'equation from ordinary methods

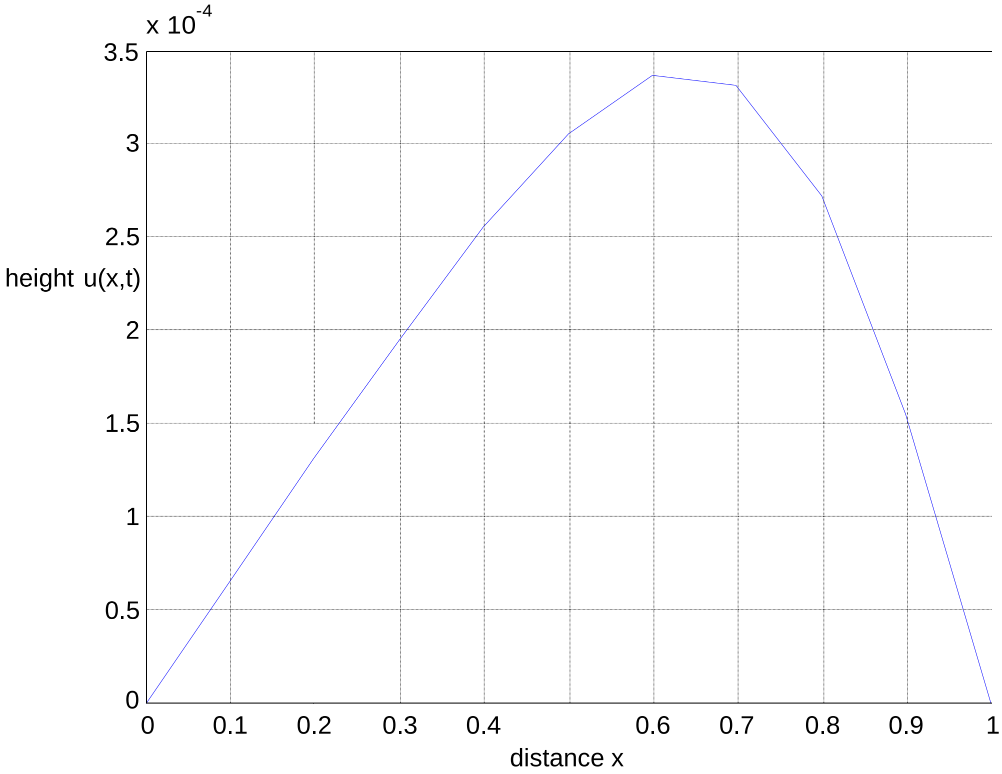


Figure 2: Exact solutions of the Burgers' equation from ordinary methods t=0.005

0

We give the two figures above because it is not clear from figure 1 where the exact solution curve is. Actually they coincide with that of CN-LF-DF scheme.

From figures 1(and 2) we realize that Du Fort and Frankel differencing improves both the efficacy of the pure Crank-Nicholson and the Hybrid Crank-Nicholson-Lax Friedrichs' methods whereas the Lax-Friedrich's discretization does otherwise. This is because the Du Fort and Frankel discretization utilizes one extra point below the point of reference, $U_{m,n} = u(mh, nk)$. The Crank-Nicholson-Lax Friedrich's- Du Fort and Frankel scheme provides the most accurate results as can be seen in the figures because of the same reason mentioned above.

The absolute errors from the hybrid schemes at t=0.005 are as given in the table 1 below.

Table 1: Absolute errors from the hybrid methods at t=0.005 × 10^{-3}

x	CN	CN-DF	CN-LF	CN-LF-DF
0	0	0	0	0
0.1	0.00053331197580	0.00065156552310	0.01050525609140 0.03243206696000	0
0.2	0.00360376977550	0.00383917119410	0.08104481096600 0.17652670848340	0
0.3	0.01291984510310	0.01326883742900	0.33507132056870	0
0.4	0.03501233165030	0.03545318492110	0.54093331616550 0.71233756528560	0
0.5	0.08068544986110	0.08121093076600	0.71094251416810 0.44779262051820	0
0.6	0.16332074460430	0.16389031622390	0	0
0.7	0.28207585348740	0.28262100928050		0
0.8	0.37571037462310	0.37614334750010		0
0.9	0.30055059783460	0.30080063107570		0
1.0	0	0		0

From Table 1 the errors involved are actually small and without loss of generality this should taken as the case for values of *t*.

All the developed hybrid schemes are actually series and the higher the number terms involved the greater is the accuracy.

Since all the hybrid methods are implicit they are actually stable.

We now give 3-D solutions obtained from the constructed hybrid methods.

Figure 3: CN-DF 3-D solution for Burgers equation from ordinary methods

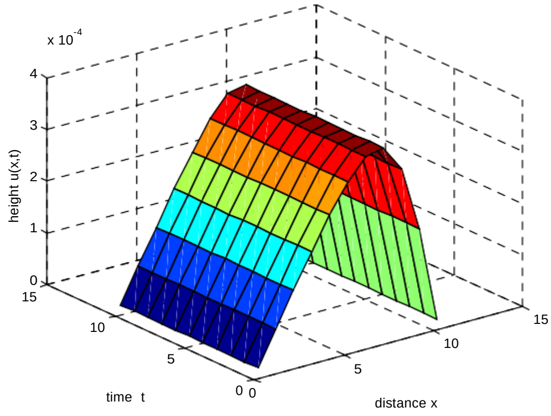


Figure 4:CN-LF 3-D solution for Burgers equation from ordinary methods

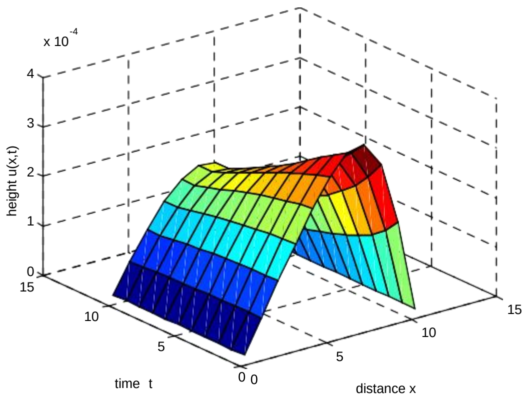
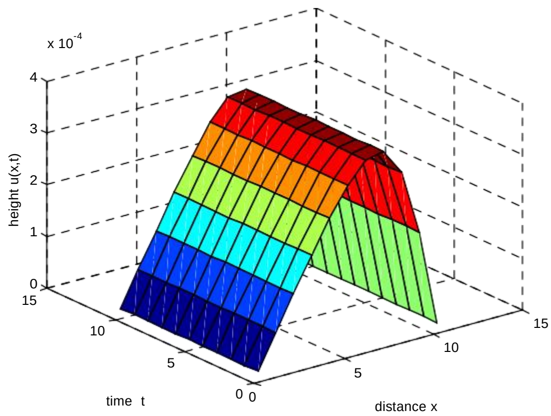


Figure 5 :CN-LF-DF 3-D solution for Burgers equation from ordinary methods



The 3-D solutions are all similar in shape but the one obtained using CN-LF-DF

(Fig. 5) provides the best picture of the distribution of fluid particles.

CONCLUSION

The Crank-Nicholson-Lax Friedrich's- Du Fort and Frankel's scheme provides the most accurate results. The Du-Fort and Frankel method increases the efficacy of both the pure Crank-Nicholson and the hybrid Crank- Nicholson-Lax-Friedrich's methods.

The Du Fort and Frankel method utilizes one grid point at the lower level of the grid point in question. The involvement of the extra grid point at lower level (to the point of reference) is responsible for the improved results of the CN- DF and CN-DF-LF methods.

The methods constructed are all based on the Crank-Nicholson method and therefore are all unconditionally stable.

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APPENDIX

The following notations are used throughout the presentations;

CN means pure Crank-Nicholson's method,

CN-LF means Crank-Nicholson- Lax-Friedrich's method,

CN-DF means Crank-Nicholson-Du Fort – Frankel's method and

CN-LF-DF means Crank-Nicholson- Lax-Friedrich-Du Fort-Frankel's method,

3-D means three- dimensional.